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## Partial differential equations for solid-liquid phase transition with fluid motion

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### Abstract

To prove the existence of a solution for nonlinear partial differential equations is the one of the interesting subject. There are various researches of tools and methods. Frequently the difficulty occurred to proof by the systemaization of equations or the nonlinearity of equations. In this paper we shall discuss about the existence problem of weak solutions for two models of nonlinear parabolic partial differential equations, which describe the solid-liquid phase transition phenomena. More precisely, these models are consisted by the Navier-Stokes equations and nonlinear heat equations as the free boundary problem. The estimate of the heat independent of the fluid motion is the key point to main theorems.

## 1 Phase transition model

The phase transition is an interesting phenomenon in which the physical state of a material dramatically changes. We are interested in the mathematical description for such a phenomenon. To do so we need to introduce some fundamental concepts of mathematical physics before exploring the phase transition. A material is a set of many atoms and molecules. But we know that averaging the phenomena in an infinitesimal area, we describe the heat conduction, the liquid flow and so on, without catching phenomena microscopically. The way for averaging is given by statistical mechanics. Nevertheless we do not intend to mention here the details of statistical mechanics. We mean by a continuum of the material which has the continuous property by averaging. The temperature or the pressure are typical quantities in continuums. By using a variable, the so-called order parameter, the change of the structure is described. How to define the order parameter is also one of the important questions. But in our setting we guarantee it by using another physical quantities, for example the difference of the density, the volume and so on. For example, the material  $\text{H}_2\text{O}$  have three kinds of phases, the ice, the water and the vapor. And their transitions occur at some critical temperatures specified the continuum. Thus in order to describe these phase transitions using quantities in continuums we apply some partial differential equations which are introduced by the various rules.

Let  $t \in [0, T]$ ,  $0 < T < +\infty$  and  $\Omega_m(t) \subset \mathbb{R}^3$  be the time dependent bounded domain with smooth boundary  $\Gamma_m(t) := \partial\Omega_m(t)$  which is known smoothly in time in the following sense:

(A1) There exists a bounded domain  $\Omega \subset \mathbb{R}^3$  with smooth boundary  $\Gamma := \partial\Omega$  such that  $\Omega_m(t) \subset \Omega$  for all  $t \in [0, T]$ . Moreover there exists a transformation  $\mathbf{y} \in \mathbf{C}^3(\bar{Q}) := (C^3([0, T] \times \bar{\Omega}))^3$  which gives a  $C^3$ -diffeomorphism  $\mathbf{y}(t, \cdot) := (y_1(t, \cdot), y_2(t, \cdot), y_3(t, \cdot))$  from  $\bar{\Omega}$  onto itself for all  $t \in [0, T]$  such that

$$\mathbf{y}(t, \overline{\Omega_m(t)}) = \overline{\Omega_m(0)} \quad \text{for all } t \in [0, T], \quad \mathbf{y}(0, \cdot) = \mathbf{I} \text{ (identity) on } \overline{\Omega_m(0)}.$$

In this paper we recall two famous models, Stefan problem and phase field equations:

## 2 Nonlinear PDEs for the phase transition

### Model S (Stefan problem)

Find  $u := u(t, x)$ : enthalpy,  $\mathbf{v} := \mathbf{v}(t, x)$ : convective vector and  $p_\ell := p_\ell(t, x)$ : pressure.

$$D_t(\mathbf{v})u - \Delta\beta(u) = f \quad \text{in } Q_m := \bigcup_{t \in (0, T)} \{t\} \times \Omega_m(t), \quad (1)$$

$$D_t(\mathbf{v})\mathbf{v} - \Delta\mathbf{v} = \mathbf{g}(\beta(u)) - \nabla p_\ell \quad \text{in } Q_\ell(u), \quad (2)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_\ell(u), \quad (3)$$

$$\mathbf{v} = \mathbf{v}_D \quad \text{in } Q_s(u) \cup S(u), \quad (4)$$

$$\frac{\partial\beta(u)}{\partial \mathbf{n}} + n_0\beta(u) = h, \quad \mathbf{v} = \mathbf{v}_D \quad \text{on } \Sigma_m := \bigcup_{t \in (0, T)} \{t\} \times \Gamma_m(t), \quad (5)$$

$$u(0) = u_0, \quad \mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \Omega_{m0} := \Omega_m(0), \quad (6)$$

where  $D_t(\mathbf{v}) := \partial/\partial t + \mathbf{v} \cdot \nabla$ ;  $f, \mathbf{g}, h, \mathbf{v}_D, u_0$  and  $\mathbf{v}_0$  are given functions;  $n_0$  is a positive constant;  $\mathbf{n} := \mathbf{n}(t, x)$  is a 3-dimensional unit outward normal vector. In Model S we define the solid-liquid interface  $S(t) := \{x \in \Omega_m(t); u(t, x) = L/2\}$  and unknown domains  $\Omega_\ell(t)$  and  $\Omega_s(t)$  by

$$\Omega_\ell(t) := \left\{x \in \Omega_m(t); u(t, x) > \frac{L}{2}\right\}, \quad \Omega_s(t) := \left\{x \in \Omega_m(t); u(t, x) < \frac{L}{2}\right\}. \quad (7)$$

Moreover we define

$$S(u) := \bigcup_{t \in (0, T)} \{t\} \times S(t), \quad Q_i(u) := \bigcup_{t \in (0, T)} \{t\} \times \Omega_i(t) \quad \text{for } i = \ell, s. \quad (8)$$

$Q_\ell(u)$  and  $Q_s(u)$  stand for the liquid and solid regions, respectively. So the equations (2) and (3) means that in the unknown liquid region, the unknown  $\mathbf{v}$  stands for the convection by the Boussinesq approximation  $\mathbf{g}(\beta(u))$  of the buoyancy force. On the other hand, the equation (4) stands for the artificial movement of the unknown solid region, which coincides with the one of  $\Omega_m(t)$ . The equation (1) is called the enthalpy formulation of the

Stefan problem, mathematically it means the weak formulation. This is the degenerate parabolic equation because  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\beta(r) := \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq L, \\ k_\ell(r - L) & \text{if } r > L, \end{cases} \quad (9)$$

where  $k_s, k_\ell$  and  $L$  are positive constants, where  $\beta(u)$  stands for the temperature. If  $0 \leq u \leq L$  then the equation (1) becomes to the ordinary differential equation, and it plays a role of the phase transition. We refer to the book of Visintin [27].

### Model P (phase field equations)

Find  $\theta := \theta(t, x)$ : temperature,  $\chi := \chi(t, x)$ : order parameter,  $\mathbf{v} := \mathbf{v}(t, x)$ : convective vector and  $p_\ell := p_\ell(t, x)$ : pressure.

$$D_t(\mathbf{v})\theta + D_t(\mathbf{v})\chi - \Delta\theta = f \quad \text{in } Q_m, \quad (10)$$

$$D_t(\mathbf{v})\chi - \Delta\chi + \chi^3 - \chi = \theta \quad \text{in } Q_m, \quad (11)$$

$$D_t(\mathbf{v})\mathbf{v} - \Delta\mathbf{v} = \mathbf{g}(\theta) - \nabla p_\ell \quad \text{in } Q_\ell(\chi), \quad (12)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_\ell(\chi), \quad (13)$$

$$\mathbf{v} = \mathbf{v}_D \quad \text{in } Q_s(\chi) \cup S(\chi), \quad (14)$$

$$\frac{\partial \theta}{\partial \mathbf{n}} = 0, \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0, \quad \mathbf{v} = \mathbf{v}_D \quad \text{on } \Sigma_m, \quad (15)$$

$$\theta(0, \cdot) = \theta_0, \quad \chi(0, \cdot) = \chi_0, \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0 \quad \text{in } \Omega_{m0}, \quad (16)$$

where  $\theta_0$  is a given function. From the stand point of the Stefan problem it is a natural setting that the sharp interface is defined by the 0-level curve of  $\chi$ . But the set  $\{(t, x) \in Q_m; \chi(t, x) = 0\}$  has the measure in general. So in Model P we image the virtual solid-liquid interface namely we call the set  $\Omega_\ell(t) := \{x \in \Omega_m(t); \chi(t, x) > 0\}$  by the liquid region, the set

$$S(t) := \left( \overline{\{x \in \Omega_m(t); \chi(t, x) = 0\}} \cap \overline{\Omega_\ell(t)} \right) \setminus \Gamma_m(t),$$

by the virtual interface and  $\Omega_s(t) := \Omega \setminus \{\Omega_\ell(t) \cup S(t)\}$  by the solid region. If  $\chi$  is continuous in  $Q_m$ , then  $\Omega_s(t)$  and  $\Omega_\ell(t)$  are open sets and  $\Omega_m(t) = \Omega_\ell(t) \cup S(t) \cup \Omega_s(t)$  for each  $t \in [0, T]$ . Moreover we define

$$S(\chi) := \bigcup_{t \in (0, T)} \{t\} \times S(t), \quad Q_i(\chi) := \bigcup_{t \in (0, T)} \{t\} \times \Omega_i(t) \quad \text{for } i = \ell, s. \quad (17)$$

We refer to the paper of Cagnalp [1], Fix [5].

We assume the following compatibility condition for Model S and P:

(A2) Given vector function  $\mathbf{v}_D \in \mathbf{C}^2(\overline{Q})$  satisfies

$$\operatorname{div} \mathbf{v}_D(t, \cdot) = 0 \quad \text{in } \Omega_m(t) \quad \text{for all } t \in [0, T], \quad (18)$$

$$\mathbf{v}_D \cdot \mathbf{n} = v_n \quad \text{on } \Sigma_m, \quad (19)$$

where  $v_n(t, \cdot)$  is the normal speed of  $\Gamma_m(t)$ .

### 3 Main theorems

In this paper we use the following notations:

$$H := L^2(\Omega), \quad Y := L^4(\Omega), \quad V := H^1(\Omega) \quad (:= W^{1,2}(\Omega)), \quad X := W^{1,4}(\Omega),$$

with the usual norms, and  $Y^*, V^*$  and  $X^*$  are the dual spaces of  $Y, V$  and  $X$ ; we denote by  $\langle \cdot, \cdot \rangle_{Y^*, Y}$ ,  $\langle \cdot, \cdot \rangle_{V^*, V}$  and  $\langle \cdot, \cdot \rangle_{X^*, X}$  the duality pairs between  $Y^*$  and  $Y$ ,  $V^*$  and  $V$  and  $X^*$  and  $X$ , respectively. Especially  $H$  is a Hilbert space with standard inner product  $(\cdot, \cdot)_H$  and we have the following relations:

$$X \subset V \hookrightarrow Y \subset H \subset Y^* \hookrightarrow V^* \subset X^*,$$

where  $\hookrightarrow$  means that the imbedding is compact. Moreover we use the following notations for vector valued function spaces:

$$\mathcal{D}_\sigma(\Omega) := \left\{ \mathbf{z} \in \mathbf{C}_0^\infty(\Omega); \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega \right\},$$

$$\mathbf{H} := \mathbf{L}_\sigma^2(\Omega), \quad \mathbf{Y} := \mathbf{L}_\sigma^4(\Omega), \quad \mathbf{V} := \mathbf{H}_\sigma^1(\Omega), \quad \mathbf{X} := \mathbf{W}_\sigma^{1,4}(\Omega),$$

where  $\mathbf{L}_\sigma^2(\Omega)$ ,  $\mathbf{L}_\sigma^4(\Omega)$ ,  $\mathbf{H}_\sigma^1(\Omega)$  and  $\mathbf{W}_\sigma^{1,4}(\Omega)$  are the closures of  $\mathcal{D}_\sigma(\Omega)$  in spaces  $\mathbf{L}^2(\Omega) := H^3$ ,  $\mathbf{L}^4(\Omega) := Y^3$ ,  $\mathbf{H}^1(\Omega) := V^3$  and  $\mathbf{W}^{1,4}(\Omega) := X^3$ , respectively. They are equipped with the usual product norms, and  $\mathbf{Y}^*, \mathbf{V}^*$  and  $\mathbf{X}^*$  are the dual spaces of  $\mathbf{Y}, \mathbf{V}$  and  $\mathbf{X}$  with duality pairs  $\langle \cdot, \cdot \rangle_{\mathbf{Y}^*, \mathbf{Y}}$  and so on. We see that  $\mathbf{H}$  is a Hilbert space with the usual inner product  $(\cdot, \cdot)_\mathbf{H}$  and the following relations hold:

$$\mathbf{X} \subset \mathbf{V} \hookrightarrow \mathbf{Y} \subset \mathbf{H} \subset \mathbf{Y}^* \hookrightarrow \mathbf{V}^* \subset \mathbf{X}^*.$$

Define  $a(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$  and for each  $t \in [0, T]$ ,  $b(t; \cdot, \cdot, \cdot) : \mathbf{Y} \times \mathbf{V} \times \mathbf{Y} \rightarrow \mathbb{R}$ ,  $c(t; \cdot, \cdot) : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$  are defined by

$$a(\mathbf{z}, \boldsymbol{\eta}) := \nu_\ell \sum_{i=1}^3 \int_\Omega \nabla z_i \cdot \nabla \eta_i dx \quad \text{for all } \mathbf{z}, \boldsymbol{\eta} \in \mathbf{V},$$

$$b(t; \mathbf{z}, \bar{\mathbf{z}}, \boldsymbol{\eta}) := \sum_{i=1}^3 \int_\Omega ((\mathbf{z} + \mathbf{v}_D(t)) \cdot \nabla \bar{z}_i) \eta_i dx \quad \text{for all } \mathbf{z}, \bar{\mathbf{z}} \in \mathbf{Y}, \boldsymbol{\eta} \in \mathbf{V},$$

$$c(t; \mathbf{z}, \boldsymbol{\eta}) := \sum_{i=1}^3 \int_\Omega (\mathbf{z} \cdot \nabla v_{Di}(t)) \eta_i dx \quad \text{for all } \mathbf{z}, \boldsymbol{\eta} \in \mathbf{H},$$

and for each  $z \in L^2(Q)$ ,  $\mathbf{g}(z) \in L^2(0, T; \mathbf{H})$  is defined by  $\mathbf{g}(z(t)) := P_L [\mathbf{g}_L(z(t))]$  with

$$\mathbf{g}_L(z) := \begin{cases} \mathbf{g}_\ell(z) - \frac{\partial \mathbf{v}_D}{\partial t} - (\mathbf{v}_D \cdot \nabla) \mathbf{v}_D + \nu_\ell \Delta \mathbf{v}_D & \text{on } Q_m, \\ 0 & \text{otherwise,} \end{cases}$$

where  $P_L : \mathbf{L}^2(\Omega) \rightarrow \mathbf{H}$  is the Leray projector. Our main theorems are given now.

**Theorem 1** Assume that (A1) and (A2) hold. Let  $f \in L^\infty(Q_m)$ ,  $h \in L^\infty(\Sigma_m)$ ,  $\mathbf{g} \in \mathbf{C}^{0,1}(\mathbb{R})$ ,  $u_0 \in L^\infty(\Omega_{m0})$  and  $\mathbf{v}_0 \in \mathbf{L}^2(\Omega_{m0})$  with  $\operatorname{div} \mathbf{v}_0 = 0$  on  $\Omega_{m0}$ . Then for any  $\varepsilon \in (0, 1]$ , there exists at least one  $\{u_\varepsilon, \mathbf{v}_\varepsilon\} \in L^\infty(Q_m) \times \mathbf{L}^2(Q_m)$  such that it satisfies the following auxiliary variational form of Model S

$$\begin{aligned} & - \int_{Q_m} u_\varepsilon D_t(\mathbf{v}_\varepsilon) \eta dx dt + \int_{Q_m} \nabla \beta(u_\varepsilon) \cdot \nabla \eta dx dt + n_0 \int_{\Sigma_m} \beta(u_\varepsilon) \eta d\Gamma_m(t) dt \\ & = \int_{Q_m} f \eta dx dt + \int_{\Sigma_m} q \eta d\Gamma_m(t) dt + \int_{\Omega_{m0}} u_0 \eta(0) dx \quad \text{for all } \eta \in W, \end{aligned} \quad (20)$$

where  $W := \{\eta \in H^1(Q_m); \eta(T) = 0\}$ ;

$$\begin{aligned} & - \int_0^T (\boldsymbol{\eta}', \mathbf{w}_\varepsilon)_{\mathbf{H}} dt + \int_0^T a(\mathbf{w}_\varepsilon, \boldsymbol{\eta}) dt + \int_0^T b(t; \mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon, \boldsymbol{\eta}) dt + \int_0^T c(t; \mathbf{w}_\varepsilon, \boldsymbol{\eta}) dt \\ & = \int_0^T (\mathbf{g}(\beta(u_\varepsilon)), \boldsymbol{\eta})_{\mathbf{H}} dt + (\mathbf{w}_0, \boldsymbol{\eta}(0))_{\mathbf{H}} \quad \text{for all } \boldsymbol{\eta} \in \mathbf{W}(\rho_\varepsilon * u_\varepsilon), \end{aligned} \quad (21)$$

where

$$\mathbf{W}(\rho_\varepsilon * u_\varepsilon) := \left\{ \boldsymbol{\eta} \in L^4(0, T; \mathbf{X}); \begin{array}{l} \boldsymbol{\eta}' \in L^2(0, T; \mathbf{H}), \quad \boldsymbol{\eta}(T, \cdot) = 0 \text{ a.e. on } \Omega, \\ \boldsymbol{\eta} = 0 \text{ a.e. on } Q \setminus Q_\ell(\rho_\varepsilon * u_\varepsilon) \end{array} \right\}.$$

**Theorem 2** Under the same assumption of Theorem 1. Let  $\{u_\varepsilon, \mathbf{v}_\varepsilon\}_{\varepsilon>0}$  be the solution constructed by Theorem 1. Then there exists a subsequence  $\{\varepsilon_n\}$  such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow +\infty$  and

$$u_{\varepsilon_n} \rightarrow u \quad \text{weakly in } L^2(Q_m),$$

$$\mathbf{w}_{\varepsilon_n} := \mathbf{v}_{\varepsilon_n} - \mathbf{v}_D \rightarrow \mathbf{w} \quad \text{weakly in } L^2(0, T; \mathbf{H}_\sigma^1(\Omega)) \quad \text{as } n \rightarrow +\infty,$$

$u$  and  $\mathbf{v} := \mathbf{w} + \mathbf{v}_D$  satisfy the variational form of (1), (5) and (6).

Above two theorem say that Model S has at least one weak solution. However, from the mathematical point of view it is difficult to handle this system because of the lack of regularity of the enthalpy  $u$ . In order to avoid this difficulty we replace the liquid region  $Q_\ell(u)$  and solid region  $Q_s(u)$  by their approximations  $Q_\ell(\rho_\varepsilon * u)$  and the class of test functions by  $\mathbf{W}(\rho_\varepsilon * u)$ . As to the vector function  $\mathbf{v}$  obtained in Theorem 2 as a limit of  $\{\mathbf{v}_{\varepsilon_n}\}$ , it is not clear whether  $\mathbf{v}$  is a solution of the variational form of the Navier-Stokes equation, because the class  $\mathbf{W}(u)$  of test functions for the limit  $u$  of  $\{u_{\varepsilon_n}\}$  is not able to be defined without the regularity of  $u$ . On the other hand, Model P may have the solution in which the order parameter  $\chi$  has the smoothness. So we can get the following existence result:

**Theorem 3** Assume that (A1) and (A2) hold. Let  $f \in L^\infty(Q_m)$ ,  $\mathbf{g} \in \mathbf{C}^{0,1}(\mathbb{R})$ ,  $\theta_0 \in H^1(\Omega_{m0})$ ,  $\chi_0 \in H^2(\Omega_{m0})$  and  $\mathbf{v}_0 \in \mathbf{L}^2(\Omega_{m0})$  with  $\operatorname{div} \mathbf{v}_0 = 0$  in  $\Omega_{m0}$ . Then there exist at least one  $\{\theta, \chi, \mathbf{v}\} \in L^\infty(Q_m) \times C(\overline{Q_m}) \times \mathbf{L}^2(Q)$  such that it satisfies the variational form of Model P.

## 4 Existence problem for Model S

In this section we shall probe Theorem 1 and 2. At first we recall some important results on the Stefan problem with prescribed convections and the Navier-Stokes equations formulated in non-cylindrical domains. Throughout this section assumptions (A1) and (A2) are always made and the same notation as in the previous section is used. Furthermore, given  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , we use the following notations:

$$Q(s_0, s) := (s_0, s) \times \Omega, \quad Q_m(s_0, s) := \bigcup_{t \in (s_0, s)} \{t\} \times \Omega_m(t),$$

$$\Sigma(s_0, s) := (s_0, s) \times \Gamma, \quad \Sigma_m(s_0, s) := \bigcup_{t \in (s_0, s)} \{t\} \times \Gamma_m(t).$$

### 4.1 Auxiliary results for the Stefan problem with convections

[A](cf. Fukao, Kenmochi and Pawłow [10]) Let  $\tilde{\mathbf{v}}$  be a 3-dimensional vector field defined on  $Q$  such that  $\tilde{\mathbf{v}} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$  and  $\tilde{\mathbf{v}} = \mathbf{v}_D$  a.e. on  $Q \setminus Q_m$ , let  $f \in L^\infty(Q_m)$  as well as  $q \in L^\infty(\Sigma_m)$ . Then, for each  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , and  $\tilde{u}_0 \in L^\infty(\Omega_m(s_0))$ , there exists at least one function  $\tilde{u}$  on  $Q_m(s_0, s)$  such that

(i)  $\tilde{u} \in L^\infty(Q_m(s_0, s))$ ,  $\beta(\tilde{u}(t)) \in H^1(\Omega_m(t))$  for a.e.  $t \in (s_0, s)$  with

$$\int_{s_0}^s |\beta(\tilde{u}(t))|_{H^1(\Omega_m(t))}^2 dt < +\infty,$$

and the 0-extension of  $\tilde{u}$  onto  $Q(s_0, s)$ , denoted by  $\tilde{u}$  again, is weakly continuous from  $[s_0, s]$  into  $H$ .

(ii)  $\tilde{u}$  satisfies the variational identity

$$\begin{aligned} & - \int_{Q_m(s_0, s)} \tilde{u} D_t(\tilde{\mathbf{v}}) \eta dx dt + \int_{Q_m(s_0, s)} \nabla \beta(\tilde{u}) \cdot \nabla \eta dx dt + n_0 \int_{\Sigma_m(s_0, s)} \beta(\tilde{u}) \eta d\Gamma_m(t) dt \\ & = \int_{Q_m(s_0, s)} f \eta dx dt + \int_{\Sigma_m(s_0, s)} q \eta d\Gamma_m(t) dt + \int_{\Omega_m(s_0)} \tilde{u}_0 \eta(s_0) dx, \end{aligned} \quad (22)$$

for all  $\eta \in H^1(Q_m(s_0))$  with  $\eta(s, \cdot) = 0$  a.e. on  $\Omega(s)$ .

(iii) Putting

$$M_1 := \max \left\{ L, |f|_{L^\infty(Q_m)}, \left| \frac{q}{n_0} \right|_{L^\infty(\Sigma_m)}, |\beta(\tilde{u}_0)|_{L^\infty(\Omega_m(s_0))} \right\},$$

we have  $|\beta(\tilde{u})|_{L^\infty(Q_m(s_0, s))} \leq M_1(1 + T)$ , and hence

$$|\tilde{u}|_{L^\infty(Q_m(s_0, s))} \leq \max \left\{ \frac{M_1}{k_s}, \frac{M_1}{k_\ell} + L \right\} (1 + T) =: M_2.$$

(iv) The following inequality holds:

$$\int_{\Omega_m(s)} \hat{\beta}(\tilde{u}(s)) dx + c_1 \int_{s_0}^s |\beta(\tilde{u})|_{H^1(\Omega_m(t))}^2 dt \leq \int_{\Omega_m(s_0)} \hat{\beta}(\tilde{u}_0) dx + M_3 T, \quad (23)$$

where  $M_3$  is a positive constant depending only on  $M_1$ ,  $M_2$ ,  $|\Omega|$ , the volume of  $\Omega$  and the maximum of  $|\Gamma_m(t)|$ , the area of  $\Gamma_m(t)$  for  $t \in [0, T]$ ,  $\hat{\beta}$  is the primitive of  $\beta$  with  $\hat{\beta}(0) = 0$ , and  $c_1$  is a positive constant satisfying that

$$c_1 |z|_{H^1(\Omega_m(t))}^2 \leq |\nabla z|_{L^2(\Omega_m(t))}^2 + n_0 |z|_{L^2(\Gamma_m(t))}^2 \quad \text{for all } z \in H^1(\Omega_m(t)) \text{ and } t \in [0, T].$$

For simplicity, we denote by  $(\text{SP}; \tilde{\mathbf{v}}, \tilde{u}_0)$  on  $[s_0, s]$  the variational problem (23), and any function  $\tilde{u}$  satisfying the above conditions (i)-(iv) is called a solution of  $(\text{SP}; \tilde{\mathbf{v}}, \tilde{u}_0)$  on  $[s_0, s]$ .

**[B]** (cf. Fukao and Kenmochi [9]) We have a sort of continuous dependence of solutions  $\tilde{u}$ , obtained by the above results, upon the convection vector field  $\tilde{\mathbf{v}}$ . Now, assume that  $\{\tilde{\mathbf{v}}_n\}$  is a bounded sequence of vector fields in  $L^\infty(0, T; \mathbf{H})$  and  $L^2(0, T; \mathbf{V})$  such that

$$\tilde{\mathbf{v}}_n = \mathbf{v}_D \quad \text{a.e. on } Q \setminus Q_m \quad \text{for all } n \in \mathbf{N},$$

$$\tilde{\mathbf{v}}_n \rightarrow \tilde{\mathbf{v}} \quad \text{weakly-}^* \text{ in } L^\infty(0, T; \mathbf{H}) \quad \text{weakly in } L^2(0, T; \mathbf{V}) \quad \text{as } n \rightarrow +\infty.$$

Let  $\tilde{u}_n$  be any solution of  $(\text{SP}; \tilde{\mathbf{v}}_n, \tilde{u}_0)$  on  $[s_0, s]$ , and let  $\tilde{u}$  is the weak- $*$  limit of a subsequence  $\{\tilde{u}_{n_k}\}$  in  $L^\infty(Q_m)$ . Then,  $\tilde{u}$  is a solution of  $(\text{SP}; \tilde{\mathbf{v}}, \tilde{u}_0)$  on  $[s_0, s]$ . Moreover the 0-extension of  $\tilde{u}_{n_k}(t)$  onto  $\mathbb{R}^3$  weakly converges in  $L^2(\mathbb{R}^3)$  to that of  $\tilde{u}(t)$  uniformly in  $t \in [s_0, s]$  and  $\beta(\tilde{u}_{n_k}) \rightarrow \beta(\tilde{u})$  in  $L^2((s_0, s) \times \mathbb{R}^3)$  as  $k \rightarrow +\infty$ .

## 4.2 Auxiliary results for the Navier-Stokes equations

We now recall an existence result for the variational problem associated with the Navier-Stokes equation formulated in non-cylindrical domains.

The solvability for the Navier-Stokes equation in non-cylindrical domains was discussed by many authors, for example Fujita and Sauer [6], Inoue and Wakimoto [11], Inoue and Ôtani [12], Kenmochi [13], Morimoto [17], Ôtani and Yamada [19] and Yamada [28]. In the existence proofs of [6] and [13] one of the main point is an extensive use of a compactness theorem of Aubin's type [24] and its extension of Kenmochi [14].

**[C]**(cf. [6], [13]) We consider the following variational problem associated with the Navier-Stokes equation in non-cylindrical domain  $Q_m(s_0, s)$ . Let  $\tilde{p} \in L^\infty(Q)$  with  $\tilde{p} \geq 0$  a.e. on  $Q$ ,  $\tilde{\mathbf{g}} \in L^2(s_0, s; \mathbf{H})$  and  $\tilde{\mathbf{w}}_0 \in \mathbf{H}$ . Then, there exists at least one function  $\tilde{\mathbf{w}}$  such that

- (i)  $\tilde{\mathbf{w}} \in L^\infty(s_0, s; \mathbf{H}) \cap L^2(s_0, s; \mathbf{V})$  with  $\tilde{\mathbf{w}} = 0$  a.e. on  $Q(s_0, s) \setminus Q_m(s_0, s)$  and  $\tilde{\mathbf{w}}$  is weakly continuous from  $[s_0, s]$  into  $\mathbf{H}$ .



(ii)  $\tilde{\mathbf{w}}$  satisfies the variational identity

$$\begin{aligned} & - \int_{s_0}^s (\boldsymbol{\eta}', \tilde{\mathbf{w}})_{\mathbf{H}} d\tau + \int_{s_0}^s a(\tilde{\mathbf{w}}, \boldsymbol{\eta}) d\tau + \int_{s_0}^s b(\tau; \tilde{\mathbf{w}}, \tilde{\mathbf{w}}, \boldsymbol{\eta}) d\tau \\ & + \int_{s_0}^s c(\tau; \tilde{\mathbf{w}}, \boldsymbol{\eta}) d\tau + \int_{s_0}^s (P_L(\tilde{p}\tilde{\mathbf{w}}), \boldsymbol{\eta})_{\mathbf{H}} d\tau = \int_{s_0}^s (\tilde{\mathbf{g}}, \boldsymbol{\eta})_{\mathbf{H}} d\tau + (\tilde{\mathbf{w}}_0, \boldsymbol{\eta}(0))_{\mathbf{H}} \quad (24) \end{aligned}$$

for all  $\boldsymbol{\eta} \in \mathbf{W}_0(s_0, s)$ ,

where for any  $s \in [0, T]$

$$\mathbf{W}_0(s_0, s) := \left\{ \begin{array}{l} \boldsymbol{\eta}' \in L^2(s_0, s; \mathbf{H}), \\ \boldsymbol{\eta}(s, \cdot) = 0 \text{ a.e. on } \Omega, \\ \boldsymbol{\eta} = 0 \text{ a.e. on } \\ Q(s_0, s) \setminus Q_m(s_0, s) \end{array} \right\}.$$

(iii) The following inequality holds:

$$\begin{aligned} & \frac{1}{2} |\tilde{\mathbf{w}}(t)|_{\mathbf{H}}^2 + c_2 \int_{s_0}^t |\tilde{\mathbf{w}}(\tau)|_{\mathbf{V}}^2 d\tau + \int_{Q(s_0, t)} \tilde{p} |\tilde{\mathbf{w}}|^2 dx d\tau \\ & \leq \frac{1}{2} |\tilde{\mathbf{w}}_0|_{\mathbf{H}}^2 + \int_{s_0}^t (\tilde{\mathbf{g}}, \tilde{\mathbf{w}})_{\mathbf{H}} d\tau \quad \text{for all } t \in [s_0, s], \end{aligned} \quad (25)$$

where  $c_2$  is a positive constant independent of  $\tilde{\mathbf{w}}_0$ ,  $\tilde{\mathbf{g}}$ ,  $\tilde{p}$  and time interval  $[s_0, s]$ .

We denote by  $(\text{NS}; \tilde{p}, \tilde{\mathbf{g}}, \tilde{\mathbf{w}}_0)$  on  $[s_0, s]$  the above variational problem associated with the Navier-Stokes equation on  $Q_m$ , and any function  $\tilde{\mathbf{w}}$  satisfying the above conditions (i)-(iii) is called a solution of  $(\text{NS}; \tilde{p}, \tilde{\mathbf{g}}, \tilde{\mathbf{w}}_0)$  on  $[s_0, s]$ . Hereafter we first construct approximate solutions of Model S, and then we prove Theorems 1 and 2 by discussing the convergence of approximate solutions.

Let  $0 = t_0^N < t_1^N < t_2^N < \dots < t_N^N = T$ , be the partition of  $[0, T]$  given by

$$t_k^N = kh_N \quad \text{for } k = 0, 1, \dots, N \quad \text{with } h_N = \frac{T}{N}.$$

We are now going to construct a sequence of approximate solutions by applying the existence results for  $(\text{SP}; \tilde{\mathbf{v}}, \tilde{u}_0)$  and  $(\text{NS}; \tilde{p}, \tilde{\mathbf{g}}, \tilde{\mathbf{w}}_0)$  on each time interval  $[t_{k-1}^N, t_k^N]$  mentioned in the previous section. Let  $f, q, u_0$  and  $\mathbf{v}_0$  be the data given for Model S. Moreover for each  $s, t \in [0, T]$ ,  $\Theta_{t,s}(\cdot)$  be the  $C^3$ -diffeomorphism in  $\Omega$  given by

$$\Theta_{t,s}(x) = \mathbf{x}(s, \mathbf{y}(t, x)) \quad \text{for all } x \in \Omega,$$

where  $\mathbf{x}(s, \cdot)$  is the inverse of  $\mathbf{y}(s, \cdot)$ ; note that  $\Theta_{t,s}$  maps  $\Omega_m(t)$  onto  $\Omega_m(s)$  for each  $s, t \in [0, T]$ . Now, for fixed positive parameters  $\varepsilon, \delta \in (0, 1]$ , let us define a set of functions  $\{u_{\varepsilon\delta,k}^N, \mathbf{w}_{\varepsilon\delta,k}^N\}_{k=1}^N$ , in the following manner (1)-(4):

- (1)  $\mathbf{w}_{\varepsilon\delta,1}^N$  is a solution of  $(\text{NS}; p_{\varepsilon\delta,0}^N/\delta, \tilde{\mathbf{g}}_{\varepsilon\delta,0}^N, \mathbf{w}_0)$  on  $[0, t_1^N]$ , where

$$\mathbf{w}_0 := \mathbf{v}_0 - \mathbf{v}_D(0),$$

$$p_{\varepsilon\delta,0}^N(t, x) := \left[ \rho_\varepsilon * \left( u_0(\mathbf{y}(t, \cdot)) - \frac{L}{2} \right) \right]^- (x),$$

and

$$\tilde{\mathbf{g}}_{\varepsilon\delta,0}^N(t, x) := \mathbf{g} \left( \beta(u_0(\mathbf{y}(t, x))) \right) \quad \text{for all } (t, x) \in Q(0, t_1^N);$$

- (2)  $u_{\varepsilon\delta,1}^N$  is a solution of  $(\text{SP}; \mathbf{v}_{\varepsilon\delta,1}^N, u_0)$  on  $[0, t_1^N]$ , where

$$\mathbf{v}_{\varepsilon\delta,1}^N := \mathbf{w}_{\varepsilon\delta,1}^N + \mathbf{v}_D \quad \text{on } Q(0, t_1^N);$$

- (3) for  $2 \leq k \leq N$ ,  $\mathbf{w}_{\varepsilon\delta,k}^N$  is a solution of  $(\text{NS}; p_{\varepsilon\delta,k-1}^N/\delta, \tilde{\mathbf{g}}_{\varepsilon\delta,k-1}^N, \mathbf{w}_{\varepsilon\delta,k-1}^N(t_{k-1}^N))$  on  $[t_{k-1}^N, t_k^N]$ , where

$$p_{\varepsilon\delta,k-1}^N(t, x) := \left[ \rho_\varepsilon * \left( u_{\varepsilon\delta,k-1}^N(t - h_N, \Theta_{t,t-h_N}(\cdot)) - \frac{L}{2} \right) \right]^- (x),$$

and

$$\tilde{\mathbf{g}}_{\varepsilon\delta,k-1}^N(t, x) := \mathbf{g} \left( \beta(u_{\varepsilon\delta,k-1}^N(t - h_N, \Theta_{t,t-h_N}(x))) \right) \quad \text{for all } (t, x) \in Q(t_{k-1}^N, t_k^N);$$

- (4) for  $2 \leq k \leq N$ ,  $u_{\varepsilon\delta,k}^N$  is a solution of  $(\text{SP}; \mathbf{v}_{\varepsilon\delta,k}^N, u_{\varepsilon\delta,k-1}^N(t_{k-1}^N))$  on  $[t_{k-1}^N, t_k^N]$ , where

$$\mathbf{v}_{\varepsilon\delta,k}^N := \mathbf{w}_{\varepsilon\delta,k}^N + \mathbf{v}_D \quad \text{on } Q(t_{k-1}^N, t_k^N).$$

Now, for each  $N \in \mathbb{N}$  we define two functions  $u_{\varepsilon\delta}^N$  on  $Q_m$  and  $\mathbf{w}_{\varepsilon\delta}^N$  on  $Q$  by

$$u_{\varepsilon\delta}^N(t, x) := u_{\varepsilon\delta,k}^N(t, x), \quad \text{if } t \in [t_{k-1}^N, t_k^N] \text{ and } x \in \Omega_m(t),$$

$$\mathbf{w}_{\varepsilon\delta}^N(t, x) := \mathbf{w}_{\varepsilon\delta,k}^N(t, x) \quad \text{if } t \in [t_{k-1}^N, t_k^N] \text{ and } x \in \Omega.$$

### 4.3 Estimates for approximate solutions

On account of our construction of  $\{u_{\varepsilon\delta,k}^N, \mathbf{w}_{\varepsilon\delta,k}^N\}$ ,

- (a)  $u_{\varepsilon\delta}^N \in L^\infty(Q_m)$ , the 0-extension of  $u_{\varepsilon\delta}^N$ , denoted by  $u_{\varepsilon\delta}^N$  again, is weakly continuous from  $[0, T]$  into  $H$ ,  $\beta(u_{\varepsilon\delta}^N(t)) \in H^1(\Omega_m(t))$  for a.e.  $t \in [0, T]$  and

$$\int_0^T |\beta(u_{\varepsilon\delta}^N(t))|_{H^1(\Omega_m(t))}^2 dt < +\infty.$$

- (b)  $\mathbf{w}_{\varepsilon\delta}^N$  is weakly continuous from  $[0, T]$  into  $\mathbf{H}$ , this implies that  $\mathbf{w}_{\varepsilon\delta}^N \in L^\infty(0, T; \mathbf{H})$ , and  $\mathbf{w}_{\varepsilon\delta}^N \in L^2(0, T; \mathbf{V})$ .

(c)  $u_{\varepsilon\delta}^N$  satisfies the following variational identity:

$$\begin{aligned}
& - \int_{Q_m} u_{\varepsilon\delta}^N D_t(\mathbf{v}_{\varepsilon\delta}^N) \eta dx dt + \int_{Q_m} \nabla \beta(u_{\varepsilon\delta}^N) \cdot \nabla \eta dx dt \\
& + n_0 \int_{\Sigma_m} \beta(u_{\varepsilon\delta}^N) \eta d\Gamma_m(t) dt = \int_{Q_m} f \eta dx dt + \int_{\Sigma_m} q \eta d\Gamma_m(t) dt + \int_{\Omega_{m0}} u_0 \eta(0) dx, \quad (26)
\end{aligned}$$

for all  $\eta \in W$  where  $\mathbf{v}_{\varepsilon\delta}^N := \mathbf{w}_{\varepsilon\delta}^N + \mathbf{v}_D$ .

(d)  $\mathbf{w}_{\varepsilon\delta}^N$  satisfies the following variational identity:

$$\begin{aligned}
& - \int_0^T (\boldsymbol{\eta}', \mathbf{w}_{\varepsilon\delta}^N)_{\mathbf{H}} dt + \int_0^T a(\mathbf{w}_{\varepsilon\delta}^N, \boldsymbol{\eta}) dt + \int_0^T b(t; \mathbf{w}_{\varepsilon\delta}^N, \mathbf{w}_{\varepsilon\delta}^N, \boldsymbol{\eta}) dt \\
& + \int_0^T c(t; \mathbf{w}_{\varepsilon\delta}^N, \boldsymbol{\eta}) dt + \frac{1}{\delta} \int_0^T (P_L(p_{\varepsilon\delta}^N \mathbf{w}_{\varepsilon\delta}^N), \boldsymbol{\eta})_{\mathbf{H}} dt = \int_0^T (\mathbf{g}_{\varepsilon\delta}^N, \boldsymbol{\eta})_{\mathbf{H}} dt + (\mathbf{w}_0, \boldsymbol{\eta}(0))_{\mathbf{H}}, \quad (27)
\end{aligned}$$

for all  $\boldsymbol{\eta} \in \mathbf{W}_0(0, T)$  where

$$\begin{aligned}
p_{\varepsilon\delta}^N(t, x) &:= \begin{cases} \left[ \rho_\varepsilon * \left( u_0(\mathbf{y}(t, \cdot) - \frac{L}{2}) \right) \right]^- (x) & \text{if } (t, x) \in Q(0, t_1^N), \\ \left[ \rho_\varepsilon * \left( u_{\varepsilon\delta}^N(t - h_N, \Theta_{t, t-h_N}(\cdot)) - \frac{L}{2} \right) \right]^- (x) & \text{if } (t, x) \in Q(t_1^N, T), \end{cases} \\
\mathbf{g}_{\varepsilon\delta}^N(t, x) &:= \begin{cases} \mathbf{g}(\beta(u_0(\mathbf{y}(t, x)))) & \text{if } (t, x) \in Q(0, t_1^N), \\ \mathbf{g}(\beta(u_{\varepsilon\delta}^N(t - h_N, \Theta_{t, t-h_N}(x)))) & \text{if } (t, x) \in Q(t_1^N, T). \end{cases}
\end{aligned}$$

Furthermore, we have the uniform estimates for  $u_{\varepsilon\delta}^N$  and  $\mathbf{w}_{\varepsilon\delta}^N$  with respect to  $\varepsilon, \delta \in (0, 1]$  and  $N \in \mathbb{N}$  which are given in the following lemmas.

**Lemma 4.** *Put*

$$M_4 := \max \left\{ L, |f|_{L^\infty(Q_m)}, \left| \frac{q}{n_0} \right|_{L^\infty(\Sigma_m)}, |\beta(u_0)|_{L^\infty(\Omega_{m0})} \right\}.$$

*Then*

$$|\beta(u_{\varepsilon\delta}^N)|_{L^\infty(Q_m)} \leq M_4 \left( 1 + \frac{T}{N} \right)^N < M_4 e^T, \quad (28)$$

$$|u_{\varepsilon\delta}^N|_{L^\infty(Q_m)} \leq \max \left\{ \frac{M_4}{k_s}, \frac{M_4}{k_\ell} + L \right\} e^T =: M_5, \quad (29)$$

for all  $\varepsilon \in (0, 1]$ ,  $\delta \in (0, 1]$  and  $N \in \mathbb{N}$ , and there is a positive constant  $M_6$  such that

$$\int_0^T |\beta(u_{\varepsilon\delta}^N(t))|_{H^1(\Omega_m(t))}^2 dt \leq M_6 \quad \text{for all } \varepsilon \in (0, 1], \delta \in (0, 1] \text{ and } N \in \mathbb{N}. \quad (30)$$

Moreover there is a positive constant  $M_7$  such that

$$\sup_{0 \leq t \leq T} |\mathbf{w}_{\varepsilon\delta}^N(t)|_{\mathbf{H}} \leq M_7, \quad |\mathbf{w}_{\varepsilon\delta}^N|_{L^2(0,T;\mathbf{V})} \leq M_7, \quad \frac{1}{\delta} \int_Q p_{\varepsilon\delta}^N |\mathbf{w}_{\varepsilon\delta}^N|^2 dx dt \leq M_7, \quad (31)$$

for all  $\varepsilon \in (0, 1]$ ,  $\delta \in (0, 1]$  and  $N \in \mathbb{N}$ .

We omit the proof, see Fukao and Kenmochi [8]. In the rest of this section we fix parameters  $\varepsilon, \delta \in (0, 1]$ . On account of the uniform estimates in Lemma 4, there is a sequence  $\{N_n\}$  of positive integers with  $N_n \uparrow +\infty$  as  $n \rightarrow +\infty$  such that  $u_{\varepsilon\delta}^{N_n} \rightarrow u_{\varepsilon\delta}$  weakly- $*$  in  $L^\infty(Q_m)$  as well as  $\mathbf{w}_{\varepsilon\delta}^{N_n} \rightarrow \mathbf{w}_{\varepsilon\delta}$  weakly in  $L^2(0, T; \mathbf{V})$  and weakly- $*$  in  $L^\infty(0, T; \mathbf{H})$  as  $n \rightarrow +\infty$ . Now, put  $\mathbf{v}_{\varepsilon\delta} := \mathbf{w}_{\varepsilon\delta} + \mathbf{v}_D$ . Then, by virtue of [B], the 0-extensions of  $u_{\varepsilon\delta}^{N_n}$  and  $u_{\varepsilon\delta}$  onto  $(0, T) \times \mathbb{R}^3$ , denoted by the same notations, satisfy that

$$u_{\varepsilon\delta}^{N_n}(t) \rightarrow u_{\varepsilon\delta}(t) \quad \text{weakly in } L^2(\mathbb{R}^3) \text{ and uniformly in } t \in [0, T], \quad (32)$$

$$\beta(u_{\varepsilon\delta}^{N_n}) \rightarrow \beta(u_{\varepsilon\delta}) \quad \text{in } L^2((0, T) \times \mathbb{R}^3) \quad \text{as } n \rightarrow +\infty. \quad (33)$$

Next, we observe from (27) with (28), (29) and (31) that  $\{d/dt \mathbf{w}_{\varepsilon\delta}^N\}$  is bounded in  $L^{4/3}(0, T; \mathbf{X}^*)$  and  $\{\mathbf{w}_{\varepsilon\delta}^N\}_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T; \mathbf{V})$ . Since  $\mathbf{V} \hookrightarrow \mathbf{H} \subset \mathbf{X}^*$ , with compact injections, it follows from the Aubin's compactness theorem that  $\{\mathbf{w}_{\varepsilon\delta}^N\}$  is relatively compact in  $L^2(0, T; \mathbf{H})$ , whence

$$\mathbf{w}_{\varepsilon\delta}^{N_n} \rightarrow \mathbf{w}_{\varepsilon\delta} \quad \text{in } L^2(0, T; \mathbf{H}) \quad \text{as } n \rightarrow \infty. \quad (34)$$

Furthermore if we put

$$\tilde{u}_{\varepsilon\delta}^{N_n}(t, x) = \begin{cases} u_0(\mathbf{y}(t, x)) & \text{if } (t, x) \in Q_m(0, t_1^{N_n}), \\ u_{\varepsilon\delta}^{N_n}(t - h_{N_n}, \Theta_{t, t-h_{N_n}}(x)) & \text{if } (t, x) \in Q_m(t_1^{N_n}, T), \end{cases}$$

we have for their 0-extensions of  $\tilde{u}_{\varepsilon\delta}^{N_n}$  and  $u_{\varepsilon\delta}$

$$\rho_\varepsilon * \tilde{u}_{\varepsilon\delta}^{N_n} \rightarrow \rho_\varepsilon * u_{\varepsilon\delta} \quad \text{uniformly on } [0, T] \times \mathbb{R}^3, \quad (35)$$

and hence

$$p_{\varepsilon\delta}^{N_n} \rightarrow p_{\varepsilon\delta} := \left[ \rho_\varepsilon * \left( u_{\varepsilon\delta} - \frac{L}{2} \right) \right]^- \quad \text{uniformly on } [0, T] \times \mathbb{R}^3, \quad (36)$$

as  $n \rightarrow \infty$ . Moreover  $\mathbf{g}(\beta(u_{\varepsilon\delta}^{N_n})) \rightarrow \mathbf{g}(\beta(u_{\varepsilon\delta}))$  in  $L^2(0, T; \mathbf{H})$  as  $n \rightarrow \infty$ . Thus we can prove the following proposition.

**Proposition 5.** *For any  $\varepsilon \in (0, 1]$  and  $\delta \in (0, 1]$  the pair of functions  $\{u_{\varepsilon\delta}, \mathbf{w}_{\varepsilon\delta}\}$  in  $L^\infty(Q_m) \times (L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H}))$  satisfies the following (37)-(40):*

$$\begin{aligned} & - \int_{Q_m} u_{\varepsilon\delta} \frac{\partial \eta}{\partial t} dx dt - \int_{Q_m} (\mathbf{v}_{\varepsilon\delta} \cdot \nabla \eta) u_{\varepsilon\delta} dx dt + \int_{Q_m} \nabla \beta(u_{\varepsilon\delta}) \cdot \nabla \eta dx dt \\ & + n_0 \int_{\Sigma_m} \beta(u_{\varepsilon\delta}) \eta d\Gamma_m(t) dt = \int_{Q_m} f \eta dx dt + \int_{\Sigma_m} q \eta d\Gamma_m(t) dt + \int_{\Omega_{m0}} u_0 \eta(0) dx, \end{aligned} \quad (37)$$

for all  $\eta \in W$ ,

$$\begin{aligned} & - \int_0^T (\eta', \mathbf{w}_{\varepsilon\delta})_{\mathbf{H}} dt + \int_0^T a(\mathbf{w}_{\varepsilon\delta}, \eta) dt + \int_0^T b(t; \mathbf{w}_{\varepsilon\delta}, \mathbf{w}_{\varepsilon\delta}, \eta) dt + \int_0^T c(t; \mathbf{w}_{\varepsilon\delta}, \eta) dt \\ & + \frac{1}{\delta} \int_0^T (P_L(p_{\varepsilon\delta} \mathbf{w}_{\varepsilon\delta}), \eta)_{\mathbf{H}} dt = \int_0^T (g(\beta(u_{\varepsilon\delta})), \eta)_{\mathbf{H}} dt + (\mathbf{w}_0, \eta(0))_{\mathbf{H}}, \end{aligned} \quad (38)$$

for all  $\eta \in \mathbf{W}_0(0, T)$ , and for a positive constant  $M_8$ , independent of  $\varepsilon \in (0, 1]$  and  $\delta \in (0, 1]$ ,

$$|\beta(u_{\varepsilon\delta})|_{L^\infty(Q_m)} \leq M_8, \quad |u_{\varepsilon\delta}|_{L^\infty(Q_m)} \leq M_8, \quad \int_0^T |\beta(u_{\varepsilon\delta}(t))|_{H^1(\Omega_m(t))}^2 dt \leq M_8, \quad (39)$$

$$\sup_{0 \leq t \leq T} |\mathbf{w}_{\varepsilon\delta}(t)|_{\mathbf{H}} \leq M_8, \quad |\mathbf{w}_{\varepsilon\delta}|_{L^2(0, T; \mathbf{V})} \leq M_8, \quad \frac{1}{\delta} \int_Q p_{\varepsilon\delta} |\mathbf{w}_{\varepsilon\delta}|^2 dx dt \leq M_8. \quad (40)$$

**Proof of Theorem 1.** Fixing  $\varepsilon \in (0, 1]$ , we discuss the convergence in  $\delta$ . Let  $\{u_{\varepsilon\delta}, \mathbf{w}_{\varepsilon\delta}\}$  be the same family as constructed in Proposition 5. By the uniform estimates (39) and (40), there are a sequence  $\{\delta_n\} \subset (0, 1]$  converging to 0 and functions  $u_\varepsilon \in L^\infty(Q_m)$  and  $\mathbf{w}_\varepsilon \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$  such that  $u_{\varepsilon\delta_n} \rightarrow u_\varepsilon$  weakly- $*$  in  $L^\infty(Q_m)$ ,  $\mathbf{w}_{\varepsilon\delta_n} \rightarrow \mathbf{w}_\varepsilon$  weakly in  $L^2(0, T; \mathbf{V})$  as  $n \rightarrow +\infty$ . We denote the 0-extensions of  $u_{\varepsilon\delta_n}$  and  $u_\varepsilon$  onto  $[0, T] \times \mathbb{R}^3$  by the same notation. In this case, just as in the previous section, it follows from the result in [B] that

$$u_{\varepsilon\delta_n}(t) \rightarrow u_\varepsilon(t) \quad \text{weakly in } L^2(\mathbb{R}^3) \text{ and uniformly in } t \in [0, T], \quad (41)$$

$$\beta(u_{\varepsilon\delta_n}) \rightarrow \beta(u_\varepsilon) \quad \text{in } L^2((0, T) \times \mathbb{R}^3) \quad \text{as } n \rightarrow +\infty, \quad (42)$$

and

$$\begin{aligned} & - \int_{Q_m} u_\varepsilon D_t(\mathbf{v}_\varepsilon) \eta dx dt + \int_{Q_m} \nabla \beta(u_\varepsilon) \cdot \nabla \eta dx dt + n_0 \int_{\Sigma_m} \beta(u_\varepsilon) \eta d\Gamma_m(t) dt \\ & = \int_{Q_m} f \eta dx dt + \int_{\Sigma_m} q \eta d\Gamma_m(t) dt + \int_{\Omega_{m0}} u_0 \eta(0) dx \quad \text{for all } \eta \in W, \end{aligned} \quad (43)$$

where  $\mathbf{v}_\varepsilon := \mathbf{w}_\varepsilon + \mathbf{v}_D$ . Moreover, by (41),  $\rho_\varepsilon * u_{\varepsilon\delta_n} \rightarrow \rho_\varepsilon * u_\varepsilon$  uniformly on  $[0, T] \times \mathbb{R}^3$  and

$$p_{\varepsilon\delta_n} \rightarrow p_\varepsilon := \left[ \rho_\varepsilon * \left( u_\varepsilon - \frac{L}{2} \right) \right]^- \quad \text{uniformly on } [0, T] \times \mathbb{R}^3 \quad \text{as } n \rightarrow +\infty. \quad (44)$$

Clearly  $Q_\ell(\rho_\varepsilon * u_\varepsilon)$  is an open subset of  $Q_m$ . Let  $(s_1, s_2) \times \omega$  be any relatively compact and open cylindrical subdomain of  $Q_\ell(\rho_\varepsilon * u_\varepsilon)$ . Then, it follows for all  $n$  sufficiently large that  $\mathbf{w}_{\varepsilon\delta_n}$  satisfies

$$\begin{aligned} & - \int_0^T (\eta', \mathbf{w}_{\varepsilon\delta_n})_{\mathbf{H}} dt + \int_0^T a(\mathbf{w}_{\varepsilon\delta_n}, \eta) dt + \int_0^T b(t; \mathbf{w}_{\varepsilon\delta_n}, \mathbf{w}_{\varepsilon\delta_n}, \eta) dt \\ & + \int_0^T c(t; \mathbf{w}_{\varepsilon\delta_n}, \eta) dt = \int_0^T (g(\beta(u_{\varepsilon\delta_n})), \eta)_{\mathbf{H}} dt, \end{aligned} \quad (45)$$

where the test function  $\eta$  is taken as follows

$$\eta \in \mathbf{W}_0(0, T) \quad \text{with } \text{supp} \eta \subset (s_1, s_2) \times \omega. \quad (46)$$

In fact, on account of (43) we have  $p_{\varepsilon\delta_n}\eta = 0$  on  $[0, T] \times \mathbb{R}^3$  for all large  $n$ , so that (45) is derived from (38) for any  $\eta$  satisfying (46). This together with estimates (39) and (40) shows that  $\{d/dt \mathbf{w}_{\varepsilon\delta_n}\}$  is bounded in  $L^{4/3}(s_1, s_2; \mathbf{W}_\sigma^{-1,4/3}(\omega))$  and  $\{\mathbf{w}_{\varepsilon\delta_n}\}$  is bounded in  $L^2(s_1, s_2; \mathbf{V}_1(\omega))$ , where  $\mathbf{V}_1(\omega)$  is the closure of  $\{\mathbf{z} \in \mathbf{C}^\infty(\omega); \text{div} \mathbf{z} = 0 \text{ in } \omega\}$  with respect to the topology of  $\mathbf{H}^1(\omega)$ . Since  $\mathbf{V}_1(\omega) \hookrightarrow \mathbf{L}_\sigma^2(\omega) \subset \mathbf{W}_\sigma^{-1,4/3}(\omega)$ , with compact injections, it follows from Aubin's compactness result again that  $\{\mathbf{w}_{\varepsilon\delta_n}\}$  is relatively compact in  $L^2(s_1, s_2; \mathbf{L}_\sigma^2(\omega))$ , which implies

$$\mathbf{w}_{\varepsilon\delta_n} \rightarrow \mathbf{w}_\varepsilon \quad \text{in } L^2(s_1, s_2; \mathbf{L}_\sigma^2(\omega)) \quad \text{as } n \rightarrow +\infty.$$

Noting that this is valid for every relatively compact and open cylindrical subdomain of the form  $(s_1, s_2) \times \omega$  in  $Q_\ell(\rho_\varepsilon * u_\varepsilon)$ , we can conclude that

$$\mathbf{w}_{\varepsilon\delta_n} \rightarrow \mathbf{w}_\varepsilon \quad \text{in } \mathbf{L}_{loc}^2(Q_\ell(\rho_\varepsilon * u_\varepsilon)) \quad \text{as } n \rightarrow +\infty, \quad (47)$$

since any compact subset of  $Q_\ell(\rho_\varepsilon * u_\varepsilon)$  can be covered by a finite number of subdomains of the form  $(s_1, s_2) \times \omega$ . Furthermore, letting  $n \rightarrow \infty$  in (38) for any  $\eta \in W_0(0, T)$  with  $\text{supp } \eta(t) \subset \Omega_\ell(t)$  for all  $t \in [0, T]$ , we see with the help of the convergences (41), (42), (44) and (47) that  $\mathbf{w}_\varepsilon$  satisfies (21). Also, we have uniform estimates

$$\begin{aligned} |\beta(u_\varepsilon)|_{L^\infty(Q_m)} &\leq M_8, \quad |u_\varepsilon|_{L^\infty(Q_m)} \leq M_8, \quad \int_0^T |\beta(u_\varepsilon(t))|_{H^1(\Omega_m(t))}^2 dt \leq M_8, \\ \sup_{0 \leq t \leq T} |\mathbf{w}_\varepsilon(t)|_{\mathbf{H}} &\leq M_8, \quad |\mathbf{w}_\varepsilon|_{L^2(0, T; \mathbf{V})} \leq M_8, \end{aligned}$$

for all  $\varepsilon \in (0, 1]$ , where  $M_8$  is the same positive constants in Proposition 5. Especially, the last estimate of (40) implies that

$$\left[ \rho_\varepsilon * \left( u_\varepsilon - \frac{L}{2} \right) \right]^- |\mathbf{w}_\varepsilon|^2 = 0 \quad \text{a.e. on } Q,$$

namely  $\mathbf{w}_\varepsilon = 0$  a.e. on  $Q_s(\rho_\varepsilon * u_\varepsilon)$ . □

**Proof of Theorem 2.** We discuss finally the convergence in  $\varepsilon$ . Let  $\{u_\varepsilon, \mathbf{w}_\varepsilon\}$  be the family constructed in Theorem 1. Then there are a sequence  $\{\varepsilon_n\}$  converging to 0 and functions  $u \in L^\infty(Q_m)$  and  $\mathbf{w} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$  such that

$$\begin{aligned} u_{\varepsilon_n} &\rightarrow u \quad \text{weakly-}^* \text{ in } L^\infty(Q_m) \quad \text{hence, weakly in } L^2(Q_m), \\ \mathbf{w}_{\varepsilon_n} &\rightarrow \mathbf{w} \quad \text{weakly in } L^2(0, T; \mathbf{V}), \quad \text{weakly-}^* \text{ in } L^\infty(0, T; \mathbf{H}) \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

By applying the result [B] we see that

$$u_{\varepsilon_n}(t) \rightarrow u(t) \quad \text{weakly in } L^2(\mathbb{R}^3) \text{ and uniformly in } t \in [0, T],$$

as well as

$$\beta(u_{\varepsilon_n}) \rightarrow \beta(u) \quad \text{in } L^2((0, T) \times \mathbb{R}^3) \quad \text{as } n \rightarrow +\infty,$$

and the limit  $u$  satisfies (20) where  $\mathbf{v} := \mathbf{w} + \mathbf{v}_D$ . Thus we obtain the conclusion. □

## 5 Existence problem for Model P

In this section we shall probe Theorem 3. The essential idea is same of the previous section. So we need to discuss about the solvability of the phase field equations with given convection in non-cylindrical domain. Throughout this section assumptions (A1) and (A2) are always made and the same notation as in the previous section is used. Furthermore, given  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , we use the same notations.

### 5.1 Auxiliary results for the phase field equations

In this subsection we discuss about the solvability of the phase field equations with given convection. Throughout this subsection, the convective vector  $\tilde{\mathbf{v}}$  is given. Now for each  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , we consider the following auxiliary system: Put  $\tilde{e} := \tilde{\theta} + \tilde{\chi}$

$$-\int_{Q_m(s_0, s)} \tilde{e} \tilde{D}_t \eta dx dt + \int_{Q_m(s_0, s)} \nabla \tilde{\theta} \cdot \nabla \eta dx dt = \int_{Q_m(s_0, s)} f \eta dx dt + \int_{\Omega_m(s_0)} \tilde{e}_0 \eta(s_0) dx, \quad (48)$$

$$\begin{aligned} & -\int_{Q_m(s_0, s)} \tilde{\chi} \tilde{D}_t \eta dx dt + \int_{Q_m(s_0, s)} \nabla \tilde{\chi} \cdot \nabla \eta dx dt + \int_{Q_m(s_0, s)} (\tilde{\chi}^3 - \tilde{\chi}) \eta dx dt \\ & = \int_{Q_m(s_0, s)} \tilde{\theta} \eta dx dt + \int_{\Omega_m(s_0)} \tilde{\chi}_0 \eta(s_0) dx, \end{aligned} \quad (49)$$

for all  $\eta \in H^1(Q_m(s_0, s))$  with  $\eta(s_0, \cdot) = 0$  a.e. on  $\Omega_m(s_0)$ , where  $\tilde{D}_t := D_t(\tilde{\mathbf{v}}) = \partial/\partial t + \tilde{\mathbf{v}} \cdot \nabla$ . Assume that  $\tilde{\theta}_0 \in H^1(\Omega_m(s_0))$ ,  $\tilde{\chi}_0 \in H^2(\Omega_m(s_0))$ . Moreover  $\tilde{\mathbf{v}} - \mathbf{v}_D \in L^2(0, T; \mathbf{V}) \cup L^\infty(0, T; \mathbf{H})$  and  $\tilde{\mathbf{v}}$  satisfies the following compatibility condition

$$\tilde{\mathbf{v}} \cdot \mathbf{n} = v_n \quad \text{on } \Sigma_m. \quad (50)$$

Then there exists uniquely  $\{\tilde{\theta}, \tilde{\chi}\} \in H^1(Q_m(s_0, s)) \times H^1(Q_m(s_0, s))$  such that

$$\begin{aligned} \sup_{t \in (s_0, s)} |\tilde{\theta}(t)|_{H^1(\Omega_m(t))} &< +\infty, \quad \int_{s_0}^s |\tilde{\theta}(t)|_{H^2(\Omega_m(t))}^2 dt < +\infty, \\ \sup_{t \in (s_0, s)} |\tilde{\chi}(t)|_{H^1(\Omega_m(t))} &< +\infty, \quad \int_{s_0}^s |\tilde{\chi}(t)|_{H^2(\Omega_m(t))}^2 dt < +\infty, \end{aligned}$$

and  $\{\tilde{\theta}, \tilde{\chi}\}$  satisfy the weak formulations (48) and (49). See Fukao [7], or more general approach by Schimperna [23]. At first we recall the important result of the imbedding theorem for spaces  $L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ . For example, Chapter 3, Section 2 in the book of Ladyženskaja, Solonnikov and Ural'ceva [15]

$$|u|_{L^r(0, T; L^q(\Omega))} \leq c_3 \left( |\nabla u|_{L^2(0, T; L^2(\Omega))}^{1-2/r} + |u|_{L^\infty(0, T; L^2(\Omega))}^{2/r} \right),$$

where  $q$  and  $r$  are arbitrary positive numbers satisfying the condition

$$\frac{1}{r} + \frac{3}{2q} = \frac{3}{4} \quad \text{with } q \in [2, 6], \quad r \in [2, +\infty], \quad (51)$$

where  $c_3$  is a positive constant. we have the following estimate especially the key point is the independence of  $\tilde{\mathbf{v}}$ .

**Lemma 6.** *For any  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , there exists a positive constant  $M_9$ , independent of  $\tilde{\mathbf{v}}$  such that*

$$\sup_{t \in (s_0, s)} |\tilde{e}(t)|_{L^2(\Omega_m(t))} + \int_{s_0}^s |\tilde{e}(t)|_{H^1(\Omega_m(t))}^2 dt \leq M_9. \quad (52)$$

$$\sup_{t \in (s_0, s)} |\tilde{\chi}(t)|_{L^2(\Omega_m(t))} + \int_{s_0}^s |\tilde{\chi}(t)|_{H^1(\Omega_m(t))}^2 dt + |\tilde{\chi}|_{L^4(Q_m(s_0, s))} \leq M_9. \quad (53)$$

**Proof.** Firstly we recall the variational formulation (49). By using Green-Stokes' formula with the help of the divergence freeness and the compatibility condition (50) we see for any  $\tau \in [s, s_0]$

$$\begin{aligned} & \int_{Q_m(s_0, \tau)} (\tilde{D}_t \tilde{\chi}) \tilde{\chi} dx dt \\ &= - \int_{Q_m(s_0, \tau)} \tilde{\chi} \frac{\partial \tilde{\chi}}{\partial t} dx dt + \int_{\partial Q_m(s_0, \tau)} \tilde{\chi}^2 (\tilde{\mathbf{n}})_t d\partial Q_m(s_0, \tau) \\ & \quad + \int_{s_0}^{\tau} \left( - \int_{\Omega_m(t)} (\tilde{\mathbf{v}} \cdot \nabla \tilde{\chi}) \tilde{\chi} dx + \int_{\Gamma_m(t)} \tilde{\chi}^2 v_n d\Gamma_m(t) \right) dt \\ &= - \int_{Q_m(s_0, \tau)} \tilde{\chi} (\tilde{D}_t \tilde{\chi}) dx dt + \int_{\Omega_m(\tau)} |\tilde{\chi}(\tau)|^2 dx - \int_{\Omega_m(s_0)} |\tilde{\chi}_0|^2 dx \\ & \quad + \int_{\Sigma_m(s_0, \tau)} \tilde{\chi}^2 (\tilde{\mathbf{n}})_t d\Sigma_m + \int_{\Sigma_m(s_0, \tau)} \tilde{\chi}^2 v_n d\Gamma_m(t) dt, \end{aligned} \quad (54)$$

where  $\tilde{\mathbf{n}}$  is the 4 dimensional normal vector outward from the lateral boundary  $\Sigma_m$  defined by

$$\tilde{\mathbf{n}} := ((\tilde{\mathbf{n}})_t, (\tilde{\mathbf{n}})_x) = \frac{1}{(|v_n|^2 + 1)^{1/2}} (-v_n, \mathbf{n}).$$

By virtue of the relation  $d\Sigma_m = (|v_n|^2 + 1)^{1/2} d\Gamma_m(t) dt$ ,

$$\int_{Q_m(s_0, \tau)} (\tilde{D}_t \tilde{\chi}) \tilde{\chi} dx dt = \frac{1}{2} \int_{\Omega_m(\tau)} |\tilde{\chi}(\tau)|^2 dx - \frac{1}{2} \int_{\Omega_m(s_0)} |\tilde{\chi}_0|^2 dx. \quad (55)$$

So taking the test function  $\eta = \chi$  in (49) with replacing  $Q_m(0, T)$  by  $Q_m(s_0, \tau)$  with  $\tau \in [s_0, s]$  we see that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega_m(\tau)} |\tilde{\chi}(\tau)|^2 dx + \int_{Q_m(s_0, \tau)} |\nabla \tilde{\chi}|^2 dx dt + \int_{Q_m(s_0, \tau)} |\tilde{\chi}|^4 dx dt \\ & \leq \int_{Q_m(s_0, \tau)} \tilde{\theta} \tilde{\chi} dx dt + \int_{Q_m(s_0, \tau)} |\tilde{\chi}|^2 dx dt + \frac{1}{2} \int_{\Omega_m(s_0)} |\tilde{\chi}_0|^2 dx. \end{aligned} \quad (56)$$



And the same way in (48) with the test function  $\eta = e$

$$\begin{aligned} & \frac{1}{2} \int_{\Omega_m(\tau)} |\tilde{e}(\tau)|^2 dx + \int_{Q_m(s_0, \tau)} |\nabla \tilde{e}|^2 dx dt \\ & \leq \int_{Q_m(s_0, \tau)} f \tilde{e} dx dt + \int_{Q_m(s_0, \tau)} \nabla \tilde{\chi} \cdot \nabla \tilde{e} dx dt + \frac{1}{2} \int_{\Omega_m(s_0)} |\tilde{e}_0|^2 dx, \end{aligned} \quad (57)$$

for all  $\tau \in [s_0, s]$ . So adding (56), (57) and using Young's inequality we see that

$$\begin{aligned} & |\tilde{e}(\tau)|_{L^2(\Omega_m(\tau))}^2 + \int_{s_0}^{\tau} |\nabla \tilde{e}(t)|_{L^2(\Omega_m(t))}^2 dt \\ & + |\tilde{\chi}(\tau)|_{L^2(\Omega_m(\tau))}^2 + \int_{s_0}^{\tau} |\nabla \tilde{\chi}(t)|_{L^2(\Omega_m(t))}^2 dt + 2 \int_{s_0}^{\tau} |\tilde{\chi}|^4 dx dt \\ & \leq 3 \int_{s_0}^{\tau} |\tilde{e}(t)|_{L^2(\Omega_m(t))}^2 dt + 3 \int_{s_0}^{\tau} |\tilde{\chi}(t)|_{L^2(\Omega_m(t))}^2 dt + |\tilde{e}_0|_{L^2(\Omega_m(s_0))}^2 \\ & + |\tilde{\chi}_0|_{L^2(\Omega_m(s_0))}^2 + |f|_{L^2(Q_m(s_0, s))}^2. \end{aligned}$$

Thanks to Gronwall's inequality we get the conclusion.  $\square$

Using the same method of Theorem 7.1 in Chapter 3, Section 7 of the book by Ladyženskaja, Solonnikov and Ural'ceva [15], we obtain the following global boundedness.

**Lemma 7.** *For any  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , there exists a positive constant  $M_{10}$  independent of  $\tilde{\mathbf{v}}$  such that*

$$|\tilde{\chi}|_{L^\infty(Q_m(s_0, s))} \leq M_{10}. \quad (58)$$

**Proof.** From the independence of  $\tilde{\mathbf{v}}$ , in order to calculate the integration by part (54) and (55), In (49) we take  $\eta = [\tilde{\chi} - M]^+$  with some large positive constant  $M$ . And then  $\tilde{\chi} - \tilde{\chi}^3 = \tilde{\chi}(1 - \tilde{\chi}^2) \leq \tilde{\chi}$  on  $\{(t, x) \in Q_m(s_0, s); \tilde{\chi}(t, x) \geq M\}$ . So thanks to the result of [15], it is enough to show that  $\tilde{\theta}$  is bounded with respect to the norm of  $L^{r^*}(s_0, s)$  as the  $L^{q^*}(\Omega_m(t))$  valued functions, where  $r^*$  and  $q^*$  are arbitrary positive numbers satisfying the condition

$$\frac{1}{r^*} + \frac{3}{2q^*} = 1 - \kappa, \quad (59)$$

with

$$q^* \in \left[ \frac{3}{2(1 - \kappa)}, +\infty \right], \quad r^* \in \left[ \frac{1}{1 - \kappa}, +\infty \right], \quad 0 < \kappa < 1.$$

By virtue of (51) and Lemma 6 with  $\kappa = 1/4$  we get the conclusion.  $\square$

**Lemma 8.** *For any  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , there exists a positive constant  $M_{11}$  depend on  $|\mathbf{v}|_{L^2(s_0, s; H^1(\Omega))}$  such that*

$$\int_{s_0}^s |\tilde{\chi}(t)|_{H^2(\Omega_m(t))}^2 dt + \sup_{t \in (s_0, s)} |\tilde{\chi}(t)|_{H^1(\Omega_m(t))} \leq M_{11}. \quad (60)$$

**Proof.** We operate  $\tilde{D}_t$  to second equation, then we get the following auxiliary equation with  $U = \tilde{D}_t \tilde{\chi}$ .

$$\tilde{D}_t U - \Delta U + 3\chi^2 U = \tilde{D}_t \tilde{e} \quad \text{in } Q_m(s_0, s), \quad (61)$$

$$\frac{\partial U}{\partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{n}}(\mathbf{v}_D \cdot \nabla \tilde{\chi}) \quad \text{on } \Gamma_m(s_0, s), \quad (62)$$

$$U(s_0) = U_{s_0} := \Delta \tilde{\chi}_0 - \tilde{\chi}_0^3 + \tilde{e}_0 \quad \text{in } \Omega_m(s_0). \quad (63)$$

This is an initial and boundary value problem of the linear heat equation with given coefficient so we have a weak solution because the right hand side makes in the following sense

$$\int_{\Omega_m(t)} \tilde{D}_t \tilde{e} \eta dx = \int_{\Omega_m(t)} \nabla \tilde{e} \cdot \nabla \eta dx - \int_{\Omega_m(t)} \nabla \tilde{\chi} \cdot \nabla \eta dx + \int_{\Omega_m(t)} f \eta dx, \quad (64)$$

for all  $\eta \in H^1(Q)$  with  $\eta(s) = 0$  a.e.  $t \in (s_0, s)$ . Moreover we assumed that  $\tilde{e}_0 \in H^1(\Omega_m(s_0))$ ,  $\tilde{\chi}_0 \in H^2(\Omega_m(s_0))$ . So there exist a positive constant  $M'_{11}$  such that  $U = \tilde{D}_t \tilde{\chi}$  satisfies the following estimate as the weak solution of the general heat equation

$$\sup_{t \in (s_0, s)} |\tilde{D}_t \tilde{\chi}(t)|_{L^2(\Omega_m(t))} \leq M'_{11}. \quad (65)$$

Finally by virtue of Lemma 6 and 7 with the equation  $\Delta \tilde{\chi} = \tilde{D}_t \tilde{\chi} - \tilde{\chi}^3 + \tilde{e}$  we get

$$\sup_{t \in (s_0, s)} |\Delta \tilde{\chi}(t)|_{L^2(\Omega_m(t))} \leq M'_{11} + M_{10} + M_9. \quad (66)$$

Thus using Lemma 6 and Lemma 7 we get the conclusion.  $\square$

**Lemma 9.** For any  $s_0, s \in [0, T]$  with  $0 \leq s_0 < s \leq T$ , there exists a positive constant  $M_{12}$  depend on  $|\tilde{\mathbf{v}}|_{L^2(s_0, s; \mathbf{V})}$  such that

$$|\tilde{e}|_{L^\infty(Q(s_0, s))} + |\tilde{e}|_{L^\infty(s_0, s; V)} + |\tilde{e}|_{L^2(s_0, s; H^2(\Omega))} \leq M_5. \quad (67)$$

**Proof.** Thanks to the assumption  $f \in L^r(0, T; L^q(\Omega))$  and the estimate (58), the same argument of Lemma 7 works to the equation of  $\tilde{e}$ . And then the estimate (60) replacing  $\tilde{\chi}$  by  $\tilde{e}$  holds. So the same argument of Lemma 8 works to the equation of  $\tilde{e}$ .  $\square$

We denote by  $(\text{PF}; \tilde{\mathbf{v}}, \tilde{\theta}_0, \tilde{\chi}_0)$  on  $[s_0, s]$  the variational problem associated with the phase field equations on  $Q_m$ , and any functions  $\{\tilde{\theta}, \tilde{\chi}\}$  satisfying the above lemmas is called a solution of  $(\text{PF}; \tilde{\mathbf{v}}, \tilde{\theta}_0, \tilde{\chi}_0)$  on  $[s_0, s]$ .

## 5.2 Proof of Theorem 3

Finally we show the key point of the proof of Theorem 3. In order to get the regularity of  $\chi$ , we consider the following imbedding theory: Let  $\bar{F}$  be a bounded set in  $L^\infty(0, T; H^2(\Omega_{m0}))$  and

$$\int_0^T |\tilde{D}_t u(t)|_{L^2(\Omega_m(t))}^2 dt < M_{13} \quad \text{for all } u := [\bar{u} \circ \mathbf{y}] \text{ with } \bar{u} \in \bar{F},$$

where  $M_{13}$  is a positive constant. Then  $\bar{F}$  is relatively compact in  $C((0, T) \times \overline{\Omega_{m0}})$ . Because in our setting the domain is time dependent, but we have the enough estimate for  $\tilde{v}$ . So the boundedness of the time derivative is coming from the one of  $\tilde{D}_t u$ . We can find the related topics in Fukao [7]. Thus we can use the same manner (a)-(d) in the previous section with replacing (SP;  $\tilde{v}, \tilde{u}_0$ ) by (PF;  $\tilde{v}, \tilde{\theta}_0, \tilde{\chi}_0$ ).

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